The z Transform

11.1 INTRODUCTION

The z-transform is a useful tool in the analysis of discrete-time signals and systems and is the discrete-time counterpart of the Laplace transform for continuous-time signals and systems. The z-transform may be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input, and design linear filters.

11.2 Z-transform

The z-transform of a discrete sequance $x(n)$,expressed as $X(z)$ is defined as unilateral transform , $X(z)$ is not concerned with the history of $x(n)$ prior to $n = 0$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

The z-transform is a function of a complex variable z

Example 11-1: find the Z-transform of the finite duration signal $x(n) = \{1,2,5,7,0,1\}$

Solution:-

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{5} x(n) z^{-n} = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

Example 11-2: Find the Z-transform of the signal $x(n) = (\frac{1}{2})^n u(n)$

Solution:

$$X(z) = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = 1 + \frac{1}{2} z^{-1} + (\frac{1}{2})^2 z^{-2} + +(\frac{1}{2})^3 z^{-3} + (\frac{1}{2})^4 z^{-4} + \cdots$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$

11.3 The properties of Z-transform

There are a number of important and useful z-transform properties. A few of these properties are described below.

- **Linearity**

If *x(n)* has a z-transform *X(z)* and if *y(n)* has a z-transform *Y(z)* then

$$w(n) = ax(n) + by(n) \overset{Z}{\longleftrightarrow} W(z) = aX(z) + bY(z)$$

- **Shifting Property**

Shifting a sequence (delaying or advancing) multiplies the z-transform by a power of z.

$$x(n-n_0) \overset{Z}{\longleftrightarrow} z^{-n_0}X(z)$$

- **Convolution Theorem**

The most important z-transform property is the convolution theorem, which states that convolution in the time domain, is mapped into multiplication in the frequency domain, that is,

$$y(n) = x(n) * h(n) \overset{Z}{\longleftrightarrow} Y(z) = X(z)H(z)$$

- **Final value theorem:**

It is regarded as property to find the final, steady state, response of a system to a step input. It is states that

If $x(n) \leftrightarrow X(z)$ then $\lim_{n\to\infty} x(n) \leftrightarrow \lim_{z\to 1}\frac{z-1}{z}X(z)$

Example 11-3: Consider the two sequences

$$x(n) = \alpha^n u(n) \qquad h(n) = \delta(n) - \alpha\delta(n-1)$$

Find the z-transform of the convolution of x*(n)* with *h(n)*

Solution: - The z-transform of *x(n)* is

$$X(z) = \frac{1}{1 - \alpha z^{-1}}$$

and the z-transform of *h(n)* is

$$H(z) = 1 - \alpha z^{-1}$$

However, the z-transform of the convolution of *x(n)* with *h(n)* is

$$Y(z) = X(z)H(z) = \frac{1}{1 - \alpha z^{-1}} \cdot (1 - \alpha z^{-1}) = 1$$

11.4 Inverse z-transform

There are several approaches available to recover a signal from the z-transform; two of them are partial fraction and long division method.

11.4.1 Partial fraction method

The algebraic method of partial fraction is used, and then the table of z-transform pairs is needed to look up the function. If the function is not listed it may be possible to express it as the sum of two or more simpler functions which do appear in the table.

The z-transform table pairs is shown below

x(n)	X(z)
$\delta(n)$	*1*
$u(n)$	$\frac{z}{z-1}$
$(a)^n u(n)$	$\frac{z}{z-a}$
Cx(n)	*cX(z)*
$x(n-n_0)u(n-n_0)$	$x(z).z^{-n0}$
$x_1(n)*x_2(n)$ for $0 \le n \le \infty$	$X_1(z).X_2(z)$

Example 11-4: Find and sketch the signal corresponding to the z-transform function $\mathrm{X(z)} = \frac{1}{1+3\mathrm{z}^{-1}+2\mathrm{z}^{-2}}$

Solution: multiply the *X(z)* by z^2 to make *X(z)* is a power of *z*

$$\mathrm{X(z)} = \frac{\mathrm{z}^2}{\mathrm{z}^2 + 3\mathrm{z} + 2}$$

The partial fraction is done for *X(z)/z* as

$$\frac{X(z)}{z} = \frac{A}{z+2} + \frac{B}{z+1}$$

A=2, B=-1

$$\mathrm{X(z)} = \frac{2z}{z+2} + \frac{-z}{z+1}$$

By matching the x(z) with the z-transform table pairs , we get

$$x(n) = 2(-2)^n u(n) - u(n)(-1)^n$$

11.4.2 The long division method

In this method we expressed the *x(z)* as a power series of z^{-1}, with coefficients equal to successive values of the time domain signal *x(n)*. Therefore if we expressed *X(z)* as a power series , we can immediately regenerate the signal.

Example 11-5: A signal has the z-transform $X(z) = \frac{1}{z(z-1)(2z-1)}$ use the long division to recover the signal

Solution: - multiplying $X(z)$ by $z^3 z^{-3}$, we get

$$X(z) = \frac{z^{-3}}{2-3z^{-1}-z^{-2}}$$

$$0.5z^{-3} + \frac{3}{4}z^{-4}$$

$$2 - 3z^{-1} - z^{-2} \;\overline{\big)\; z^{-3} \qquad\qquad\qquad}$$

$$\mp z^{-3} \pm \frac{3}{2}z^{-4} \pm \frac{1}{2}z^{-5}$$

$$\frac{3}{2}z^{-4} + \frac{1}{2}z^{-5}$$

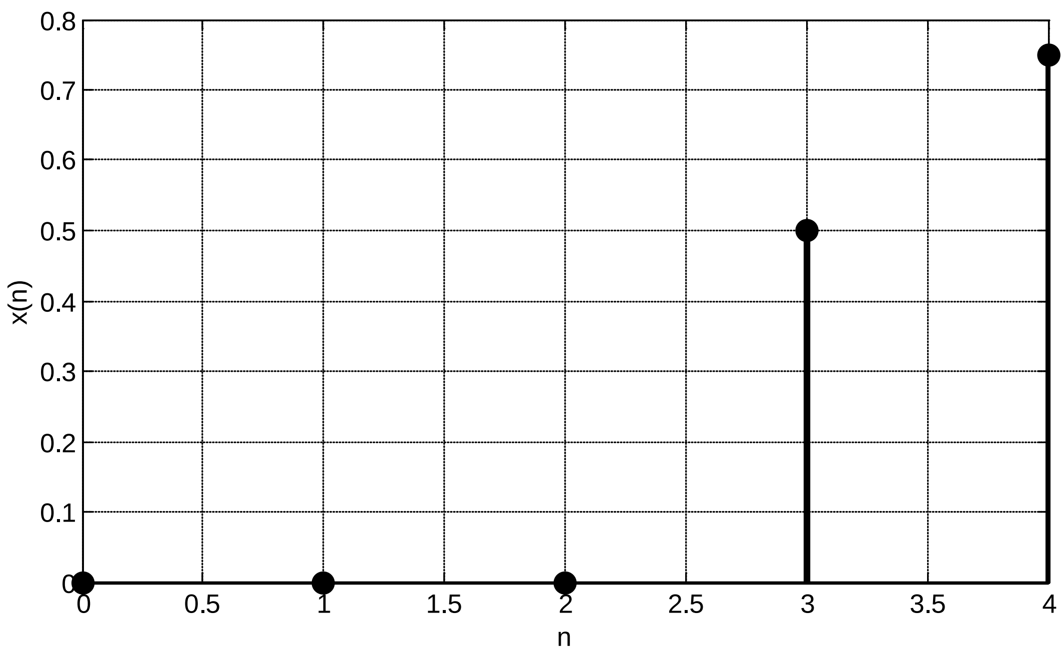
$$\mp \frac{3}{2}z^{-4} \mp \frac{9}{4}z^{-5} \pm \frac{3}{4}z^{-6}$$

The required power series $x(z) = 0.5z^{-3} + 3/4z^{-4}$

Since $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$

The coefficient of *X(z)* are equal to the sample value of *x(n)*

x(0)=0 , x(1)=0, x(2)=0, x(3)=0.5 , x(1)=3/4, and so on



11.5 poles and zeros on the z-plane and stability

The z-transform used to described a real signal or any LTI system , it is always a rational function of the frequency variable z . it can be written as the ratio of numerator and denominator polynomials in z

$$X(z) = \frac{N(z)}{D(z)}$$

Where X(z) represents an input or output signal , or the transfer function of a processor

The X(z) can be expressed in the form

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)(z-z_2)(z-z_3)\ldots}{(z-p_1)(z-p_2)(z-p_3)\ldots}$$

The constants $z_1, z_2, z_3\ldots$ are called the zero of X(z), because they are the values of z for which X(z) is zero. Conversely $p_1, p_2, p_3\ldots$ are known as the poles of X(z), giving values of z for which X(z) tends to infinity. When the time function is real, then the poles and zeroes are themselves either real, or occur in complex conjugate pairs. A very useful representation of a z-transform is obtained by plotting its poles and zeros in the complex plane referred as a z-plane. Note that a zero is shown as an open circular symbol, and a pole as a cross. The z-plane represent as a circle of unit radius centered at the z-plane origin as shown in Fig(11-1).

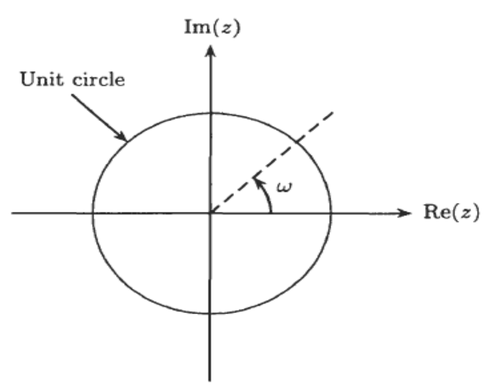


Fig (11-1) The unit circle in the complex z-plane.

One of the most characteristics of the z-plane is that the region of the filter stability is mapped to the inside of the unit circle on the z-plane. Given the H(z) transfer function of a digital filter , we can examine that functions pole locations to determine filter stability. If all poles are located inside the unit circle, the filter will be stable. On the other hand, if any pole is located outside the unit circle, the filter will be unstable.

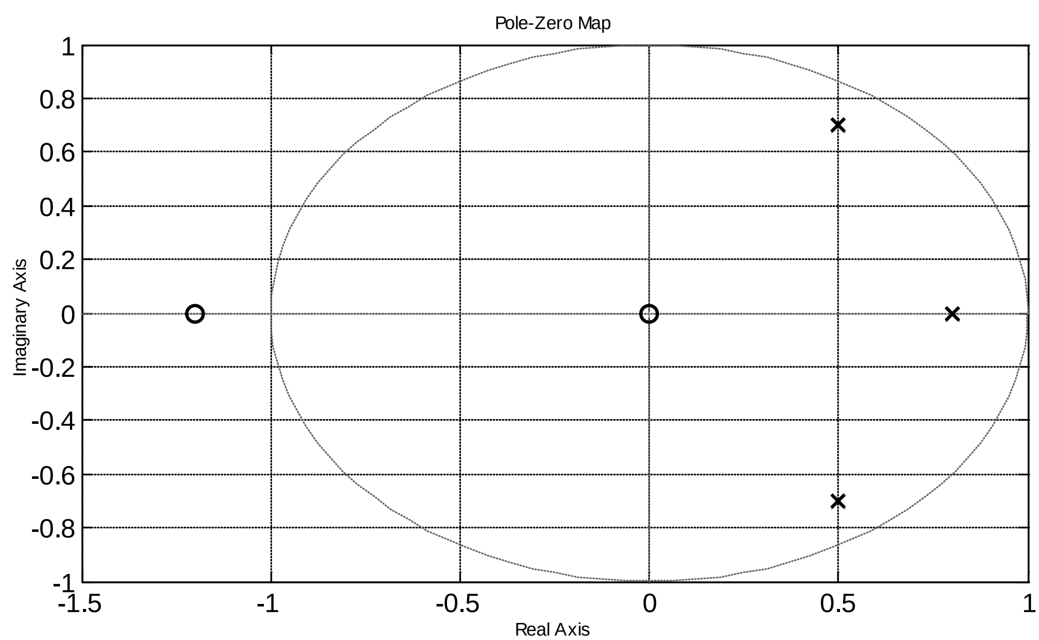
Example11-6:- plot the z-plane poles and zeros of the following z-transforms $X(z) = \frac{z^2(z+1.2)}{(z-0.5+j0.7)(z-0.5-j0.7)(z-0.8)}$

Solution:

Zeros: *z=0, z=-1.2*

Poles: *p=0.5-j0.7, p=0.5+j0.7, z=0.8*

The system is stable because all its poles lie inside the unit circle



11.6 Solving the linear constant coefficient difference equation:

The primary use of the one-sided z-transform is to solve linear constant coefficient difference equations that have initial conditions. Most of the properties of the one-sided z-transform are the same as those for the two-sided z-transform. One that is different, however, is the shift property. Specifically, if x(n) has a one-sided z-transform $X_1(z)$, the one side z-transform of x(n-1) is

$$x(n-1) \stackrel{Z}{\longleftrightarrow} z^{-1}X_1(z) + x(-1)$$

Example 11-7: Consider the linear constant coefficient difference equation

$$y(n) = 0.25y(n-2) + x(n)$$

Find the solution to this equation (find the output of this system) assuming that $x(n) = \delta(n-1)\ with\ y(-1) = y(-2) = 1$

Solution: the one-sided z-transform of *y(n - 2)* is

$$\sum_{n=0}^{\infty} y(n-2)z^{-n} = y(-2) + y(-1)z^{-1} + \sum_{n=0}^{\infty} y(n)z^{-n-2} = y(-2) + y(-1)z^{-1} + z^{-2}Y_1(z)$$

Therefore, taking the z-transform of both sides of the difference equation, we have

$$Y_1(z) = 0.25[y(-2) + y(-1)z^{-1} + z^{-2}Y_1(z)] + X_1(z)$$

Where $X_1(z) = z^{-1}$.Substituting for $y(-1) = y(-2) = 1$ and solving for $Y_1(z)$, we have

$$Y_1(z) = \frac{1}{4}\frac{1+5z^{-1}}{1-\frac{1}{4}z^{-2}}$$

To find y(n), note that $Y_1(z)$ may be expanded by partial fraction as

$$Y_1(z) = \frac{\frac{11}{8}}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{9}{8}}{1 + \frac{1}{2}z^{-1}}$$

Therefore

$$y(n) = \left[\frac{11}{8}\left(\frac{1}{2}\right)^n - \frac{9}{8}\left(-\frac{1}{2}\right)^n\right]u(n)$$

Home works:

1. Consider a LTIS described by the following z-transform transfer function

   $H(z) = \frac{1}{z(z-1)(2z-1)}$, Find and sketch the impulse response

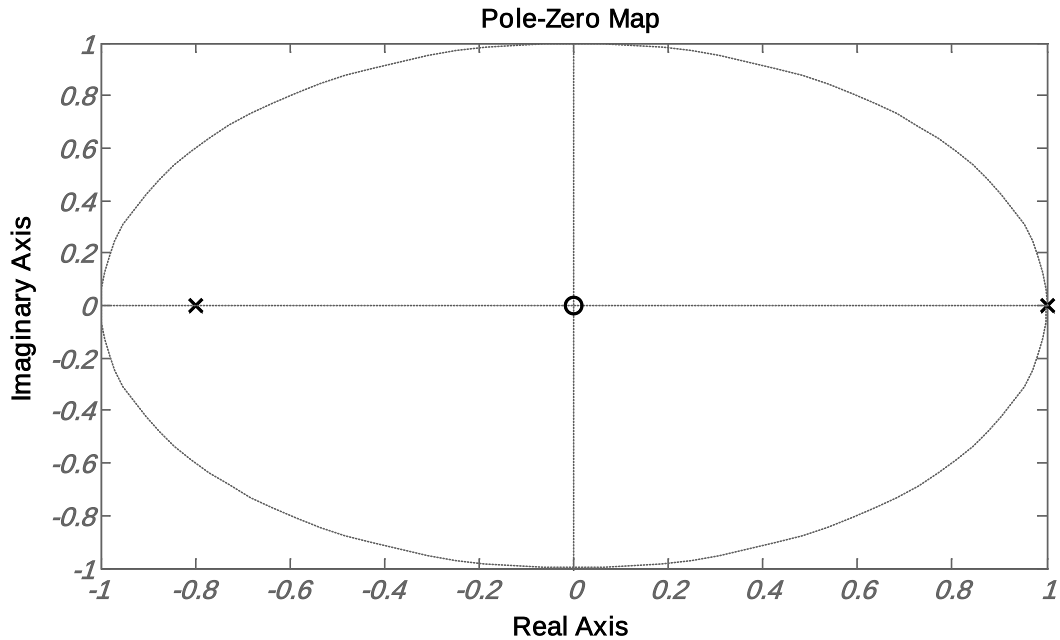
2. Two digital signal are

   $x_1(n) = \delta(n) - \delta(n-2) + \delta(n-3)$

   $x_2(n) = 2\delta(n-1) - \delta(n-2) - \delta(n-3)$

   Write the respective z-transform of $X_1(z), X_2(z)$. Convolve the two signals to form the third signal $x_3(n)$ and show that its z-transform equals $X_1(z).X_2(z)$.

3. Find the signal corresponding to the pole-zero configuration shown below



4. Find the final value of the step response of a filter described by the following difference equation $y(n)-0.8y(n-1)=x(n)$