Digital Filter Design

12.1 INTRODUCTION

An LTIS digital filter can be uniquely identified in the time domain by its impulse response *h(n)* (where *n* is an integer index). Alternatively, the LTIS digital filter can be uniquely characterized in the frequency domain by its frequency response *H(w)* (where *w* is a real-valued frequency variable in radians), which is also the Discrete-Time Fourier Transform (DTFT) of the sequence *h(n)*.

LTIS digital filters are of two main types: Finite-duration Impulse Response (FIR) filters for which the impulse response *h(n)* is non-zero for only a finite number of samples, and Infinite-duration Impulse Response (IIR) filters for which *h(n)* has an infinite number of non-zero samples. In the FIR case, the samples of the sequence *h(n)* are commonly referred to as the filter coefficients; for the IIR case, the filter coefficients include feedback terms in a difference equation.

12.2 Filter specification

The term filter is used to describe a device that discriminates what passes through it, according to the attribute of the objective applied at its input. Filters are usually classified according to their frequency domain characteristics as low-pass filter, high-pass filter, band-pass filter, and band stop filter. The Ideal frequency responses of these filters are shown in figure 12-1. These ideal filters have a unity gain in passband and zero gain in stopband.

The general filter design problem can be briefly stated as follows. Given some ideal frequency response *DH(w)* find a realizable IIR or FIR digital filter whose frequency response ,*H(w)* approximates *DH(w).* Since the frequency response of a digital filter is always periodic in the frequency variable *w* with a period of 2π, the design need only be specified for one period, usually, over the frequency region $[-\pi , \pi]$.

The simplest case is that of an ideal low-pass digital filter with zero phase, whose frequency response can be expressed as:

$$D(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Where w_c is the cutoff frequency corresponding to the location of a sharp cutoff edge, as shown in Fig. 12.1(a). In this case, the frequency response, *H(w)* is real-valued and, therefore, corresponds also to the magnitude response of the filter (since the phase is zero).

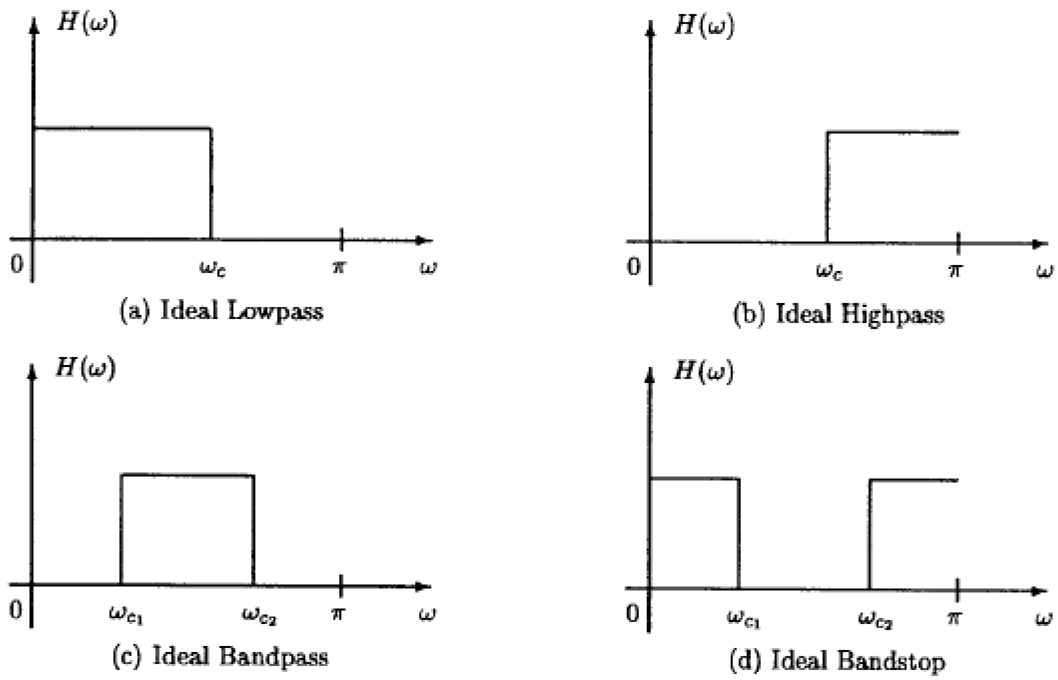


Figure 12-1 Common ideal digital filter types.

These ideal filters have frequency responses with sharp cutoff edges (discontinuities) and cannot be implemented directly. They must be approximated with a realizable system—the sharp cutoff edges need to be replaced with *transition bands* in which the designed frequency response would change smoothly in going from one band to the other. So, *design templates* need to be provided where the sharp cutoff edges are replaced with non-zero width transition bands located around the ideal cutoff edges. A typical design template for a lowpass filter is shown in Fig. 12.2, where:

- w_p is the passband cutoff frequency.
- w_s is the stopband cutoff frequency. The cutoff frequency w_c is usually taken to be midway between the passband and stopband cutoff frequencies.
- The open interval (w_p, w_s) is the transition band of width $\Delta w_t = w_s - w_p$.
- δ_p is known as the passband ripple and is the maximum allowable error in the passband.
- δ_s is known as the stopband ripple and is the maximum allowable error in the stopband.

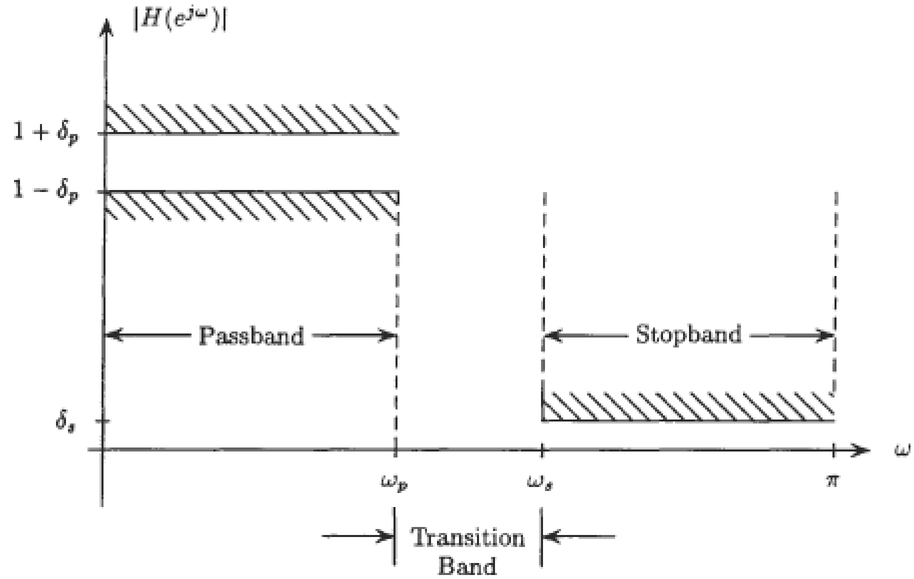


Figure 12-2 Filter specifications for a low-pass filter,

The objective of filter design then is to find a realizable FIR or IIR filter whose frequency response *H(w)* approximates the specified design constraints given by the design template. Ideally, the filter design process would make each of the following parameters as small as possible: δ_p, δ_s , Δw_t.

12.3 FIR filter design using windows

The Fourier transform method is widely used for the design of nonrecursive digital filter. It is used to specify a filter with any desired form of frequency response. The frequency response of an Nth-order causal FIR filter is

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n)e^{-jn\omega}$$

Then the corresponding unit sample response is obtained by the inverse Fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_0^{2\pi} H_d(w)\, e^{jwn}\, dw$$

The design of an FIR filter involves finding the coefficients h(n) that result in a frequency response that satisfies a given set of filter specifications.

Because $h_d(n)$ will generally be infinite in length, it is necessary to find an FIR approximation to $H_d(w)$. With the window design method, the unit sample response $h_d(n)$ must be truncated at some point; this process is equivalent to multiplying it by a finite length window function

$$h(n) = h_d(n)w(n)$$

Where w(n) is a finite-length window that is equal to zero outside the interval $0 \le n \le N$. and *it* is symmetric about its midpoint.

The multiplication in time domain is equivalent to frequency domain convolution

$$H_A(w) = H_D(w) \circledast W(w)$$

There are many different types of windows that may be used in the window design method, two of them are:

- Rectangular window: the window length $=N+1$.

$$w(n) = \begin{cases} 1 & 0 \le n \le N \\ 0 & \text{else} \end{cases}$$

- Hamming window

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & \text{else} \end{cases}$$

The windows length is N+1.The hamming window has best performance and it is widely used in recursive filter design

As the length N of the window increases, results in decreases the transition width between passbands and stopbands. This relationship is given approximately by

$$N\Delta f = c$$

where Δf is the transition width, and c is a parameter that depends on the window, for rectangular window $c = 0.9$,while for hamming window $c = 3.3$.

Note that:

- *Increasing the windows length can decreases the transition width between passband and stopband, but the ripples amplitude is decreases by changing the windows shape, it is independent of the windows length.*
- *The window design method requires that the filter be designed to the tightest tolerances in all the bands by selecting the smallest transition width and the smallest ripple.*

Example 12-1:- Design an ideal, causal FIR low pass filter of length 7 with specifications

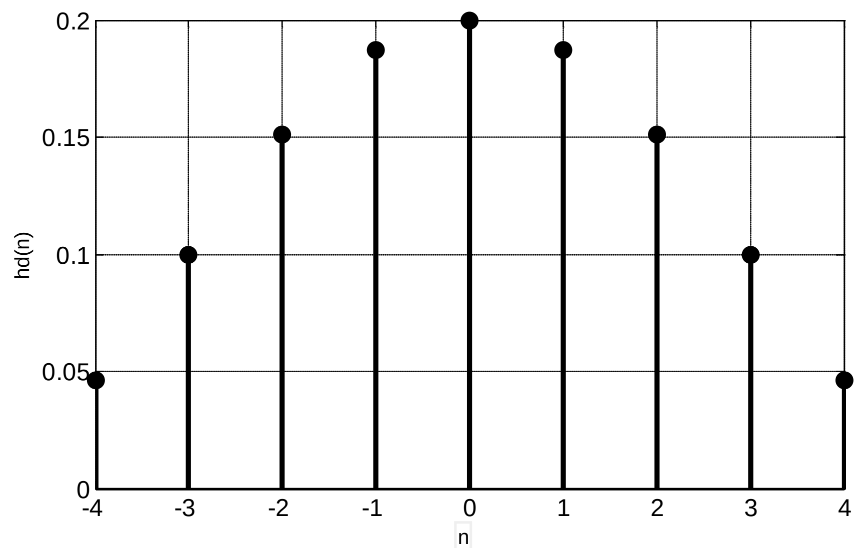
$H(w) = \begin{Bmatrix} 1 & \frac{-\pi}{5} \le w \le \frac{\pi}{5} \\ 0 & otherwise \end{Bmatrix}$, sketch its impulse response

Solution:- the cutoff frequency of low pass filter is $\frac{\pi}{5}$, using the Fourier transform method

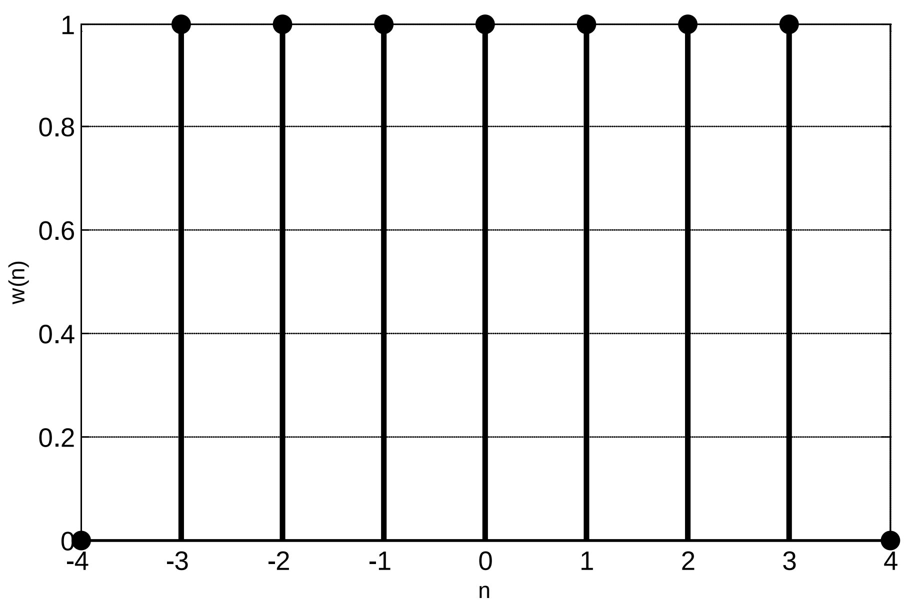
$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(w).e^{jwn}\,dw = \frac{1}{2\pi}\int_{\frac{-\pi}{5}}^{\frac{\pi}{5}} 1.e^{jwn}\,dw = \frac{1}{n\pi}sin\frac{\pi n}{5}$$

h(0) is obtained by hopitals' rule, $h(0) = \left.\frac{\frac{d}{dn} sin\frac{\pi n}{5}}{\frac{d}{dn}\pi n}\right|_{n=0} = 0.2$

n	h(n)
0	0.2
1	0.187
2	0.151
3	0.1
-1	0.187
-2	0.151
4	0.046

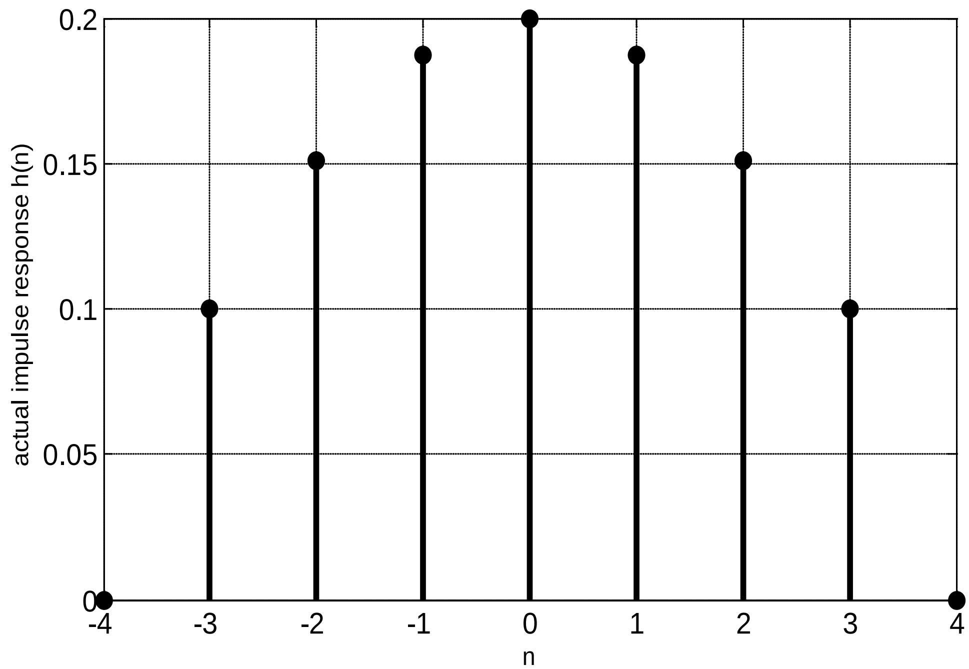


The impulse response must be truncated to obtain FIR filter of length 7, this is obtained by multiplying $h_d(n)$ by rectangular window of length 7

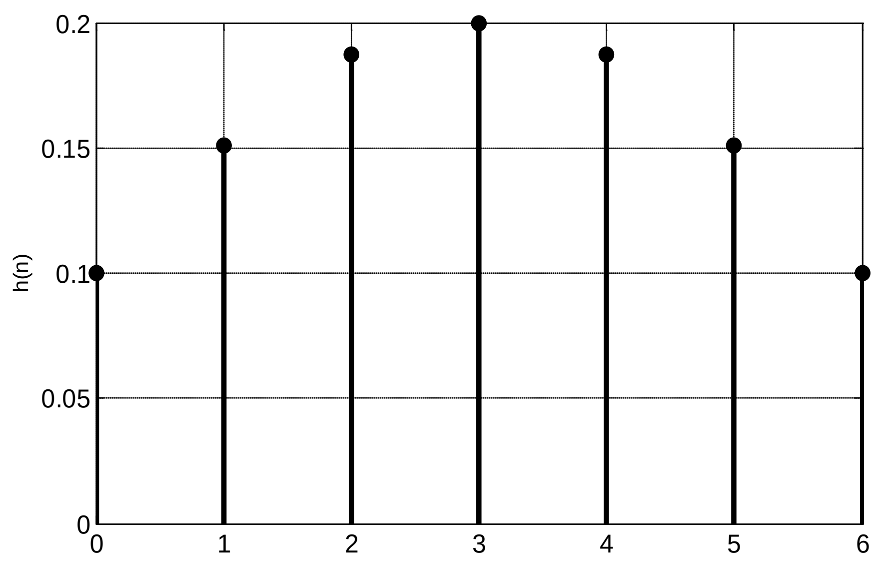


Where $w(n) = \begin{cases} 1 & 0 \le n \le 6 \\ 0 & otherwise \end{cases}$

$h(n) = h_d(n)w(n)$



Then shift h(n) to begin at n=0 to produce causal linear phase FIR filter



The causal FIR filter is

$$h(n) = 0.1\delta(n) + 0.15\delta(n-1) + 0.187\delta(n-2) + 0.2\delta(n-3) + 0.187\delta(n-4) + 0.15\delta(n-5) + 0.1\delta(n-6)$$

The difference equation of this filter is

$$y(n) = 0.1x(n) + 0.15x(n-1) + 0.187x(n-2) + 0.2x(n-3) + 0.187x(n-4) + 0.15x(n-5) + 0.1x(n-6)$$

Note: the small ripples amplitude is difficult to see in linear plot, so the algorithmic plot is used to measure the spectral magnitude in decibel.

For example let the magnitude of frequency response at specific $(w_0)=G$

$|H(w_0)| = G$

Then the algorithmic scale will be

$|H(w_0)|_{db} = 20logG$

Example 12-2: Consider the following specifications for a low-pass filter:

$$0.99 \le |H(e^{j\omega})| \le 1.01 \qquad 0 \le |\omega| \le 0.3\pi$$
$$|H(e^{j\omega})| \le 0.01 \qquad 0.35\pi \le |\omega| \le \pi$$

Design a linear phase FIR filter to meet these specifications using the window design method. Using both rectangular and Hamming windows. Find

1. Stopband attenuation (dB)
2. transition band
3. cutoff frequency of low pass filter
4. The length of window in both cases

Solution: the window design method generally produces a filter with ripples of the same amplitude in the passband and stopband. Therefore, because the passband and stopband ripples in the filter specifications are the same, we only need to be concerned about the stopband ripple requirement.

Amplitude ripple $\delta_1=\delta_2=0.01$

Attenuation in stopband =attenuation in passband=*20log δ_1=-40db*

The transition width Δw =0.35π-0.3π=0.05π rad/sec

$\Delta f = \frac{\Delta w}{2\pi} = 0.025\ Hz$

Case 1: using rectangular window

$N\Delta f = c \Rightarrow N = 36$

The rectangular window will be of length 37 (*0→36)*

The low pass filter cutoff frequency w_c = 0.325π (the midpoint of the transition band)

$w_c = w_p + \frac{\Delta w}{2} = 0.025\pi + 0.3\pi = 0.325\pi$

The desired impulse response of a low pass filter is

$h_d(n) = \frac{\sin(w_c n)}{n\pi} = \frac{\sin(0.325\pi n)}{n\pi}$

a delay of **α** = N/2 = 18, the unit sample response is

$h_d(n) = \frac{\sin(0.325\pi(n-18))}{(n-18)\pi}$

$$h_A(n) = h_d(n).w(n) = \begin{cases} \frac{\sin(0.325\pi(n-18))}{(n-18)\pi}.1 & 0 \le n \le 36 \\ 0 & \text{otherwise} \end{cases}$$

Case 2: using hamming window

$N\Delta f = c \Rightarrow N = \frac{3.3}{0.025} = 132$

The rectangular window will be of length 133 (*0→132)*

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right) & 0 \le n \le N \\ 0 & \text{else} \end{cases}$$

The desired impulse response of a low pass filter is

$h_d(n) = \frac{\sin(w_c n)}{n\pi}$

By shifting $h_d(n)$ by N/2 = 66, then the actual unit sample response is

$$h_d(n) = \frac{\sin\left(0.325\pi(n-66)\right)}{(n-66)\pi} \cdot \left(0.54 + 0.46\cos\frac{2\pi n}{N}\right) \qquad for\ 0 \le n \le 132$$

Homework

1. Consider the following specifications for a bandpass filter:

$$\begin{aligned} |H(e^{j\omega})| &\le 0.01 & 0 &\le |\omega| \le 0.2\pi \\ 0.95 \le |H(e^{j\omega})| &\le 1.05 & 0.3\pi &\le |\omega| \le 0.7\pi \\ |H(e^{j\omega})| &\le 0.02 & 0.8\pi &\le |\omega| \le \pi \end{aligned}$$

   Design a linear phase FIR filter to meet these specifications using a hamming window. Find attenuation (db) in passband

2. Sketch Hamming window for *N=7*
3. Use inverse Fourier transform to find the impulse response of an ideal , zero phase, low pass filter with cutoff frequency 0.4π.