7.1 The Fourier transform of aperiodic discrete sequence:

Most practical digital signals are aperiodic, they are not strictly repetitive. The Fourier transform is the relevant technique of the Fourier analysis applied to aperiodic sequences.

The Fourier transform of a non periodic signal is

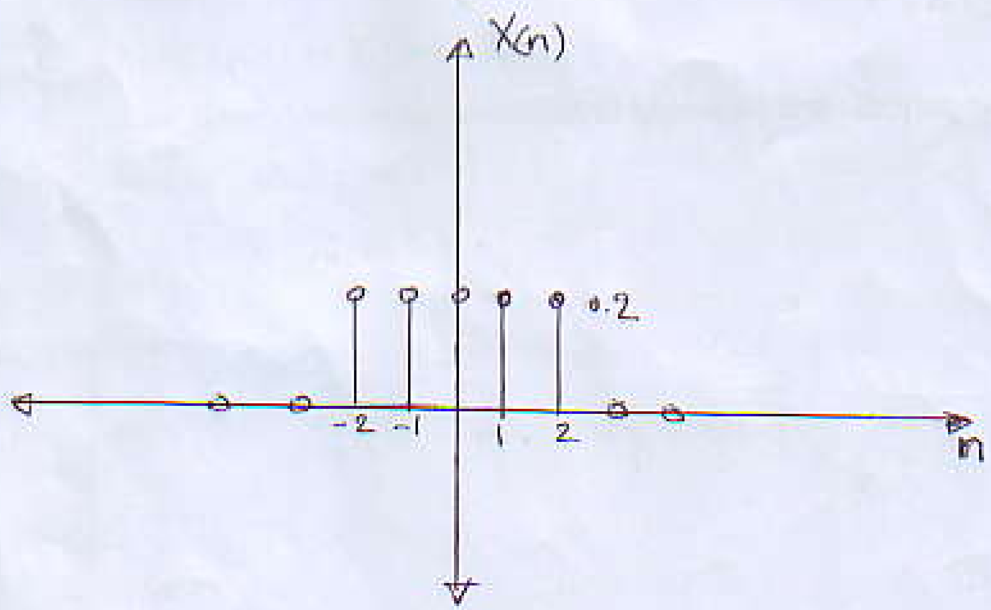
$$X(w) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$$

(This equation is called the analysis equation, it is show how the aperiodic signal can be expressed in term of imaginary exponential (or sine's and cosines) . The signal *x(n)* can be synthesized or regenerated from its spectrum *x(w)* as

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(w)e^{jwn}dw$$

Note that: The spectrum of a digital signal is always repetitive unlike that of an analog signal.

Example (7-1): Find the Fourier transform of an aperiodic signal shown below and sketch its magnitude over the range $-2\pi < w < 2\pi$



Solution:-

The Fourier transform of a non periodic signal is

$X(w) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}$

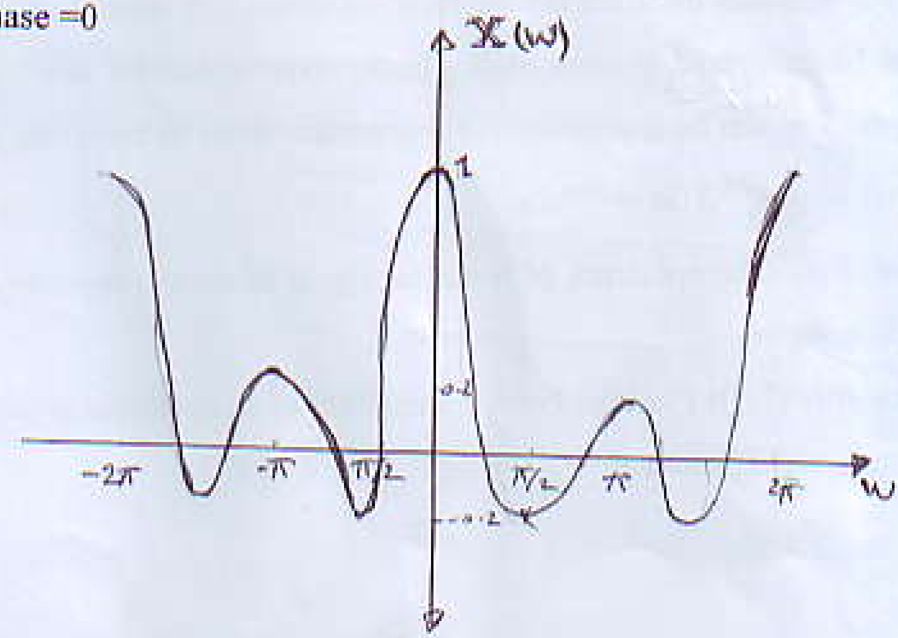
$X(w) = \sum_{n=-2}^{2} x(n)e^{-jwn} = x(-2)e^{2jw} + x(-1)e^{1jw} + x(0)e^{0} + x(1)e^{-1jw} + x(2)e^{-2jw}$

$X(w) = 0.2 + 0.4cos2w + 0.4cosw$

The spectrum is a real function, then

$X(w) = |X(w)|$, phase =0

w	X(w)
0	1
π/2	-0.2
-π/2	-0.2
π	0.2
-π	0.2
2π	1
-2π	1



This spectrum is periodic and its period is 2π

7.2 The properties of the Fourier transform

The most important properties of the transform are linearity, time shift, and convolution

- Linearity

  If $x_1(n) \leftrightarrow X_1(w)$

  $x_2(n) \leftrightarrow X_2(w)$

  Then

  $ax_1(n) + bx_2(n) \leftrightarrow aX_1(w) + bX_2(w)$

- Time shift

  If $x(n) \leftrightarrow X(w)$

  Then $x(n - n0) \leftrightarrow X(w).e^{-jwn0}$

  The time shift is equivalent to multiply the spectrum by an imaginary exponential in the frequency domain

- Convolution

  If $x_1(n) \leftrightarrow X_1(w)$

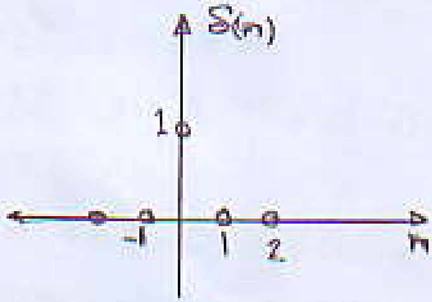
  $x_2(n) \leftrightarrow X_2(w)$

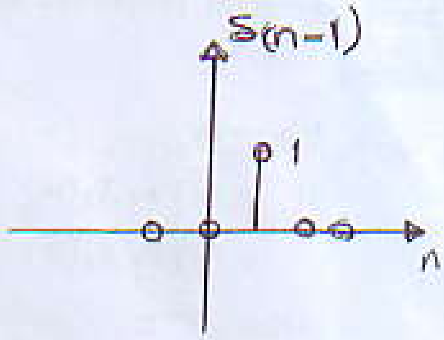
  Then

  $x_1 \otimes x_2(n) \leftrightarrow X_1(w).X_2(w)$

  The time domain convolution is equivalent to frequency response multiplication

Example (7-2): Find the spectrum of an isolated unit impulse at n=0 and delayed impulse as shown below



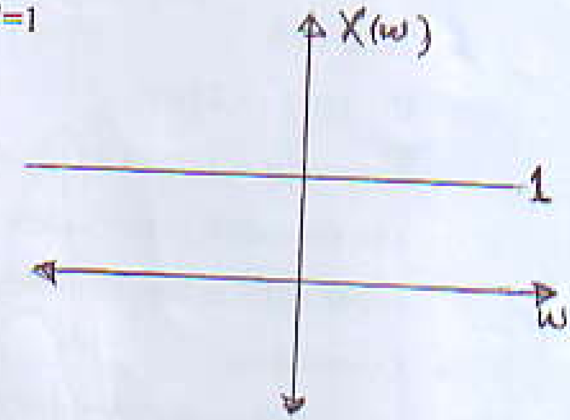


Solution:-

a- $X(w) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn} = x(0)e^{0}=1$

$X(w)$ is a real function

$|X(w)| = 1, \Phi = 0$



the spectrum of an impulse contain an equal amount of all frequencies, so this spectrum is called white just as white light contains an equal mixture of all colors.

b- The delayed impulse according to the shift property

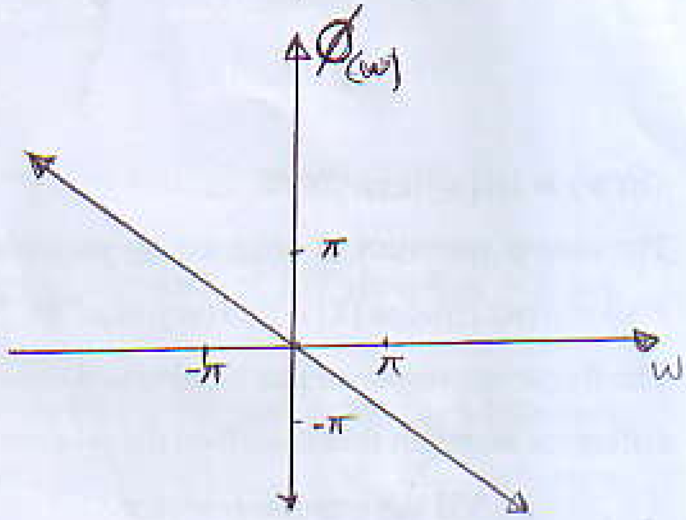
$x(n-n0) = X(w).e^{-jwn0}$

$\delta(n-1) \leftrightarrow X_2(w) = 1.e^{-jw.1} = cosw - jsinw$

The magnitude is still unity, but there is a phase shift proportional to frequency

$|X_{2(w)}| = \sqrt{(cosw)^2 + (\sin w)^2}=1$, $\Phi=\tan^{-1}\frac{-sinw}{cosw} = \tan^{-1} tanw = -w$

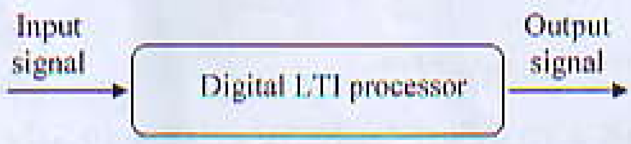
w	Φ
-π	π
π	-π
0	0



7.3 Frequency response of LTI system

The Fourier transform is used to investigate the spectra of aperiodic digital signals. There is another useful application its ability to describe the frequency domain performance of LTI systems.

The relationship of the output for a LTI processor in both time and frequency domain is shown below



Time domain $x(n)$ $h(n)$ $y(n)=x(n)\otimes h(n)$

Frequency domain $X(w)$ $H(w)$ $Y(w)=X(w).H(w)$

In the time domain, the input signal $x(n)$ is convolved with the impulse response $h(n)$to produce the output signal . In frequency ~~response~~ domain, the output signal spectrum $Y(w)$ is the product of the input spectrum $X(w)$ and a function $H(w)$. $H(w)$ is known as the systems frequency response.

The spectrum of the input and the frequency response may be a complex number, it can write as a polar form as

$X(w) = |X(w)| exp^{j\Phi_x(w)}$

$H(w) = |H(w)| exp^{j\Phi_H(w)}$

The output spectrum is obtained by multiple the magnitude and add the phases

$Y(w) = X(w).H(w) = |X(w)||H(w)| exp^{j(\Phi_X + \Phi_H)}$

The frequency response can be obtained either via the impulse response or via the difference equation that described the processor.

- Via the impulse response

  The impulse response $h(n)$ and the frequency response $H(w)$ are a Fourier transform pair

  $H(w) = \sum_{n=-\infty}^{\infty} h(n) e^{-jwn}$

Example (7-3):- Consider the impulse response of a LTIS is

$h(n) = 0.5\delta(n) + 0.25\delta(n-1) + 0.125\delta(n-2) + \cdots$, find the frequency response of this system

Solution:-

$$H(w) = \sum_{n=-\infty}^{\infty} h(n) e^{-jwn}$$

$$= 0.5 + (0.5)^2 e^{-jw} + (0.5)^3 e^{-2jw} + (0.5)^4 e^{-3jw} + \cdots$$

$$= 0.5[1 + 0.5e^{-jw} + (0.5)^2 e^{-2jw} + (0.5)^3 e^{-3jw} + (0.5)^4 e^{-4jw} + \cdots$$

$$= \frac{0.5}{1 - 0.5e^{-jw}}$$

- Via The general form of the causal LTI difference equation is

  $\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$

By taking the Fourier transform to both sides, according to the property of linearity and time shift, we get

$\sum_{k=0}^{N} a_k y(w) e^{-jwk} = \sum_{k=0}^{M} b_k x(w) e^{-jwk}$

$$\frac{y(w)}{x(w)} = H(w) = \frac{\sum_{k=0}^{M} b_k e^{-jwk}}{\sum_{k=0}^{N} a_k e^{-jwk}}$$

This equation allow to find the frequency response of any recursive and non recursive LTI processor

Example (7-4): consider the system described by the following difference equation $y(n) = 1.5y(n-1) - 0.85y(n-2) + x(n)$, find the frequency response of this system

Solution: $y(n) - 1.5y(n-1) + 0.85y(n-2) = x(n)$

$$a_0 = 1, a_1 = -1.5\,, a_2 = 0.85, b_0 = 1$$

$$H(w) = \frac{\sum_{k=0}^{M} b_k e^{-jwk}}{\sum_{k=0}^{N} a_k e^{-jwk}}$$

$$= \frac{\sum_{k=0}^{0} b_k e^{-jwk}}{\sum_{k=0}^{2} a_k e^{-jwk}} = \frac{b_0 e^{-jw.0}}{a_0 + a_1 e^{-jw.1} + a_2 e^{-jw.2}} = \frac{1}{1 - 1.5e^{-jw.1} + 0.85e^{-jw.2}}$$

Home work :-

1- A linear time invariant system described by the following difference equation

$$y(n) = a y(n-1) + b x(n) \qquad 0 < a < 1$$

(a) Determine the magnitude and phase of the frequancy response $H(w)$ of the system

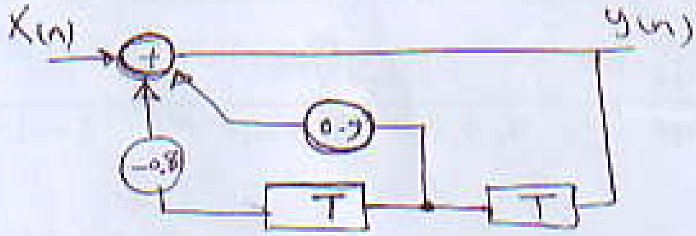
(b) choose the parameter b so that the maximum value of $|H(w)|$ is unity and sketch $|H(w)|$ and $\phi_{H(w)}$ for $a = 0.9$

2. Find the spectrum $X(\omega)$ for the following aperiodic signals

ⓐ $x(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$

ⓑ $x(n) = u(n+3) - u(n-4)$

3- A bandpass filter is shown below, find an expression for its frequancy response magnitude $|H(\omega)|$ and sketch the function in the range $0 < \omega < \pi$



Fourier analysis for signal and system

6.1 Introduction

A practical signal may always be analysis into set of sine and cosine components with appropriate amplitudes and frequencies. If the signal is an even function (symmetrical about the time origin), it contains only cosine. If it is an odd function (antisymmetrical about the time origin), it contains only cosine. If the signal is periodic, its frequency component is harmonically related. The spectrum has a number of discrete spectral lines, and is called a line spectrum it is mathematically by a Fourier series .when the signal is non-repetitive (a periodic) , it can be expressed as the finite sum (integral)of sinusoids or exponentials which are not harmonically related. The corresponding spectrum is continuous and is described mathematically by the Fourier transform.

The frequency–domain approach is used for three main reasons:

- Sinusoidal and exponential signals occur in the natural world, even when the signal is not of this type, it can be analyzed into component frequencies. The response of an LTI processor to each such component is quite simple. it can only alter the amplitude and phase , not the frequency.
- If an input signal is described by its frequency spectrum and an LTI processor by its frequency response then the output signal spectrum is found by multiplication which is simpler to perform than the time domain convolution.
- The design of DSP algorithms and systems starts with a frequency domain specification .Because it specifies which frequency ranges in an input signal are to be enhanced, and which are suppressed (such as low pass filter, high pass filter ...etc)

6.2 Fourier transform

Fourier transformation of a signal gives its spectrum. A complementary process, inverse Fourier transformation allows regenerating the signal.

The Fourier transform of a continuous signal *x(t)* is defined as

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Where *X(w)* is the spectrum of *x(t)*it is may be real or complex, *w* is the radian frequency, *t*=time, *x(t)* =continues time signal .

The inverse Fourier transform is defined as

$$x(t) = \int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Example (6-1): - Find and sketch the spectrum of the following continuous signal $x(t) = e^{-at}u(t)$ choose a=1

Solution:-

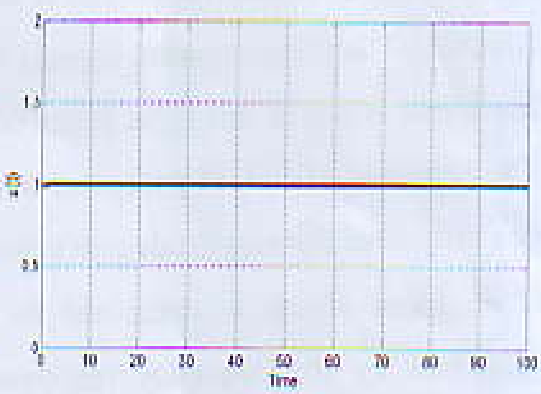
$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

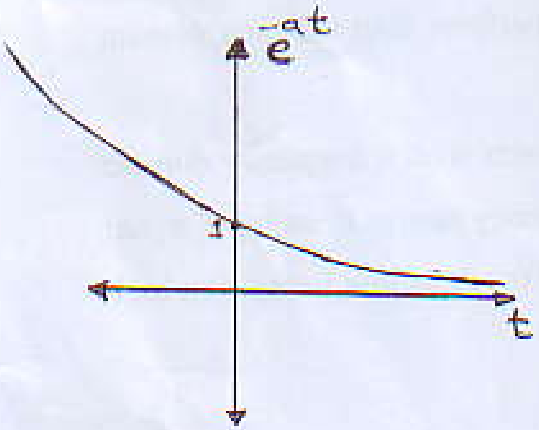
$$= \int_{0}^{\infty} e^{-at}.1.e^{-jwt}dt$$

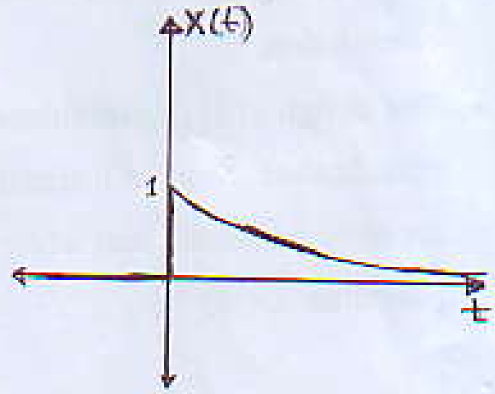
$$= \int_{0}^{\infty} e^{-(a+jw)t}dt$$

$$X(w) = \frac{1}{a+jw}$$

$$|X(w)| = \frac{1}{\sqrt{a^2+w^2}}, \quad \Phi(w) = -\tan^{-1}\frac{W}{a}$$

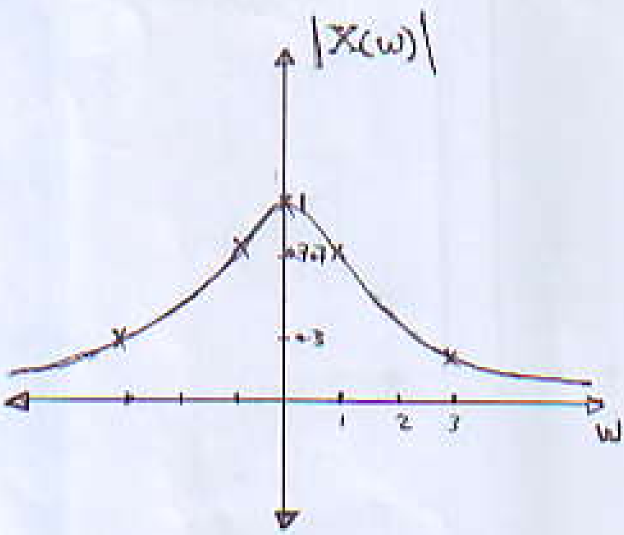


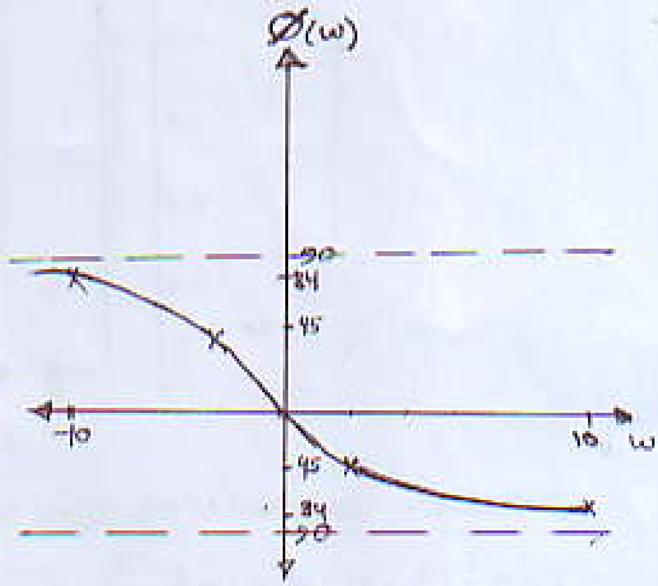




w	$\|X(w)\|$
0	$\frac{1}{a} = 1$
1	$\frac{1}{\sqrt{2} = 0.707}$
-1	0.707
3	0.3
-3	0.3

w	$\Phi(w)$
0	0
1	-45
-1	45
-10	84
10	-84
∞	90

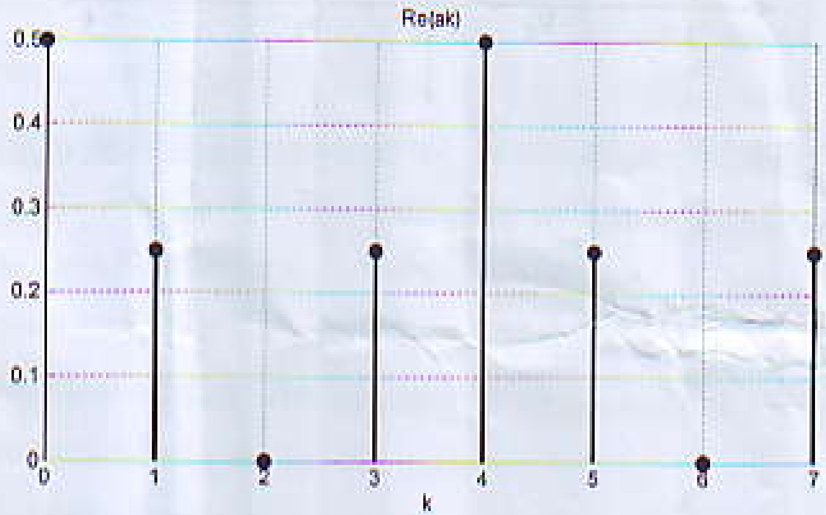


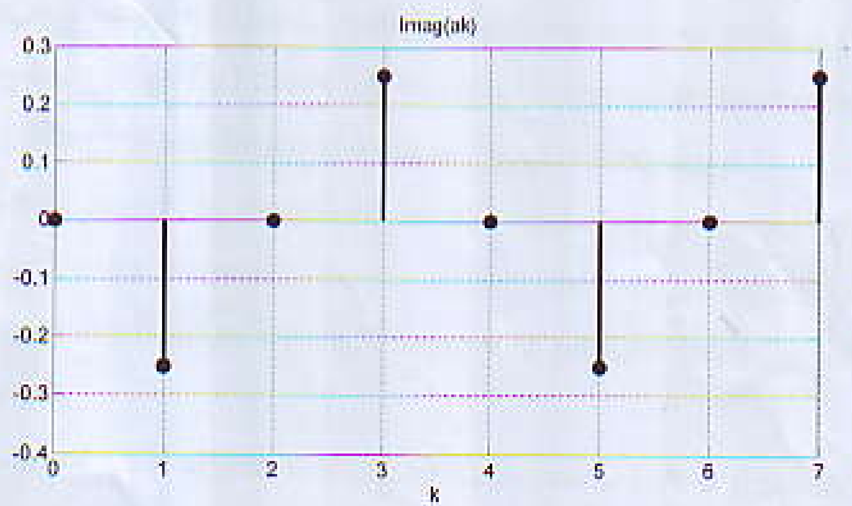


6.3 The Discrete Fourier series

A periodic digital signal can be represented by a Fourier series, it has a line spectrum. The coefficients of its line spectrum indicate the amount of various frequencies contained in the signal. They may be found using the equation $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp\left(-\frac{j2\pi kn}{N}\right)$

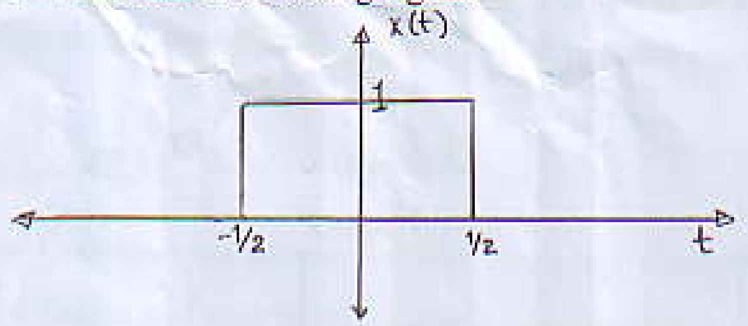
The period of $x(n)$ is 4 sample per period ,so its Fourier series must repeat every four harmonics. The real and imaginary part of the Fourier series coefficients are shown below





Homework

1- Find the spectrum of the following signal



2- Find the spectral coefficient a_k for the following periodic digital signals

a. $x(n) = 1 + sin\frac{n\pi}{4} + 2cos\frac{n\pi}{2}$

b. $x(n) = \cos\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)$