Differential Equations

A differential equation is an equation that involves one or more derivatives, or differentials. There are two types of differential equations ordinary and partial.

Order

The order of differential equation is the highest order derivative that occurs in the equation.

Degree

The exponent of the highest power of the highest order derivative.

Ex1:

$\frac{dy}{dx} = 5x + 3$ 1st order-1st degree

Ex2:

$(\frac{d^3y}{dx^3})^2 + (\frac{d^2y}{dx^2})^5$ 3rd order-2nd degree

Ex3:

$4\frac{d^3y}{dx^3} + sinx\left(\frac{d^2y}{dx^2}\right) + 5xy = 0$ 3rd order-1st degree

Exercise: Find the order and degree of these differential equations.

1. $\frac{dy}{dx} + cosx = 0$ ans:1st order-1st degree
2. $3dx+4y^2dy=0$ ans:1st order-1st degree
3. $\frac{d^2y}{dx^2} + y = y^2$ ans:2nd order-1st degree
4. $(y'')^2+2y'=x^2$ ans:2nd order-2nd degree
5. $y'''+2(y'')^2=xy$ ans:3rd order-1st degree

Solution

The solution of the differential equation in the unknown function y and the independent variable x is a function y(x) that satisfies the differential equation.

Ex: Show that $y=c_1 \sin 2x+c_2 \cos 2x$ is a solution of $y''+4y=0$

sol:

$y=c_1 \sin 2x+c_2 \cos 2x$

$y'= 2c_1 \cos 2x. - 2c_2 \sin 2x.$

$y''=-4 c_1 \sin 2x-4 c_2 \cos 2x$

$-4 c_1 \sin 2x-4 c_2 \cos 2x+4(c_1 \sin 2x+c_2 \cos 2x)=0$

$\therefore$ y is a solution

Note:

The solution in example above is called general solution since it's contain an arbitrary constant c_1 and c_2, i.e. the general solution of differential equation is the set of all solution, and the particular solution is any one of these solutions.

Exercise:

1. Show that $y=3e^{2x}-e^{-2x}$ is a solution to $y''-4y=0$
2. Determine whether $y(x)= 2e^{-x}+xe^{-x}$ is a solution of $y''+2y'+y=0$
3. Determine whether $y= x^2-1$ is a solution of $(y')^4+y^2=-1$

Ordinary Differential Equations:

Ordinary Differential Equations are equations involve derivatives.

A. First Order D.Eqs.

1- Variable Separable.
2- Homogeneous.
3- Linear.
4- Exact.

1- Variable Separable:

A first order D.Eq. can be solved by integration if it is possible to collect all y terms with dy and all x terms with dx, that is, if it is possible to write the D.Eq. in the form

$$f(x)dx + g(y)dy = 0$$

then the general solution is:

$\int f(x)dx + \int g(y)dy = c$, where c is an arbitrary constant.

Ex.1:

Solve $\dfrac{dy}{dx} = e^{x+y}$

Sol.:

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\frac{dy}{e^y} = e^x dx$$

$$\int e^{-y} dy = \int e^x \cdot dx$$

$$-\int e^{-y} \cdot (-dy) = \int e^x dx \quad \Rightarrow \quad -e^{-y} = e^x + c$$

Ex.2:

Solve $(1+x)\dfrac{dy}{dx} = x(y^2+1)$

Sol.:

$$\int \frac{dy}{(y^2+1)} = \int \frac{x}{x+1} dx$$

$$\tan^{-1} y = \int dx - \int \frac{1}{x+1} dx$$

$$\tan^{-1} y = x - \ln|x+1| + c$$

$$\frac{x}{x+1} = \frac{x+1-1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = 1 - \frac{1}{x+1}$$

Ex.3:

Solve $\frac{dy}{dx}=(y-x)^2 \quad \cdots(1)$

Sol.:

$$put \quad y-x=u\,,\ \frac{dy}{dx}-1=\frac{du}{dx} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{du}{dx}+1 \quad \ldots\ldots (2)$$

From (1) & (2)

$$\frac{du}{dx}+1=u^2 \quad \Rightarrow \quad \int\frac{du}{u^2-1}=\int dx$$

$$\frac{1}{(u-1)(u+1)}=\frac{A}{u-1}+\frac{B}{u+1} \quad \text{(partial fraction)}$$

$$1=Au+A+Bu-B$$

$$0=A+B$$

$$1=A-B \quad , \quad A=\frac{1}{2} \quad , \quad B=\frac{-1}{2}$$

$$\therefore \int\left[\frac{1/2}{u-1}+\frac{-1/2}{u+1}\right]du=\int dx$$

$$\frac{1}{2}\left[\ln(u-1)-\ln(u+1)\right]=x+c$$

$$\frac{1}{2}\ln\frac{u-1}{u+1}=x+c$$

$$\frac{u-1}{u+1}=e^{2x+c}$$

Exercise: Separate the variables and solve**.**

1. $x(2y-3)dx+(x^2+1)dy=0$ ans: $(x^2+1)(2y-3)=c$
2. $dy=e^{x-y}\,dx$ ans: $e^y=e^x+c$
3. $\sin x\frac{dy}{dx}+\cosh 2y=0$ ans: $\sinh 2y-2\cos x=c$
4. $xe^y dy+\frac{x^2+1}{y}dx=0$ ans: $e^y(y-1)+\frac{x^2}{2}+\ln|x|=c$
5. $\sqrt{2xy}\frac{dy}{dx}=1$ ans: $\frac{\sqrt{2}}{3}y^{\frac{3}{2}}=x^{\frac{1}{2}}+c$

2- Homogeneous:

Some times a D.Eq. which variables can't be separated can be transformed by a change of variables into an equation which variables can be separated. This is the case with any equation that can be put into form: $\frac{dy}{dx} = f(\frac{y}{x})$ …(1)

Such an equation is called homogenous.

Put $\frac{y}{x} = u \quad \Rightarrow \quad y = ux\,, \ \frac{dy}{dx} = u + x \cdot \frac{du}{dx}$

and (1) becomes

$x \cdot \frac{du}{dx} + u = f(u)$, this equation can be solved by separation of variables.

$$\frac{dx}{x} + \frac{du}{u - f(u)} = 0$$

Ex.1:

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

Sol.:

$$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{y}{x}} \qquad \text{Put} \quad \frac{y}{x} = u \quad \Rightarrow \frac{dy}{dx} = \frac{1+u^2}{u} = f(u)$$

$$\frac{dx}{x} + \frac{du}{u - f(u)} = 0 \ \Rightarrow \frac{dx}{x} + \frac{du}{u - \frac{1+u^2}{u}} = 0 \Rightarrow \frac{dx}{x} + \frac{du}{\frac{u^2 - 1 - u^2}{u}} = 0$$

$$\int \frac{dx}{x} + \int -u \cdot du = 0 \Rightarrow \ln x - \frac{u^2}{2} = c \quad \Rightarrow \quad \frac{y^2}{2x^2} = \ln x + c$$

Ex.2: Solve the homogenous D.Eq. x dy- 2ydx=0

Sol.: xdy=2ydx $\Rightarrow \frac{dy}{dx} = \frac{2y}{x} = 2u = f(u)$

$$\frac{dx}{x} + \frac{du}{u - 2u} = 0 \ \Rightarrow \frac{dx}{x} + \frac{du}{-u} = 0 \Rightarrow \ln|x| - \ln|u| = c$$

$$\frac{x}{u} = c \Rightarrow \frac{x^2}{y} = c \Rightarrow y = \frac{x^2}{c} \quad or \quad y = kx^2$$

Exercise: Show that the following differential equations are homogenous and solve.

1. $(x^2+y^2)dx+xy\,dy=0$ ans: $x^2(x^2+2y^2)=c$
2. $x^2dy+(y^2-xy)dx=0$ ans: $y = \frac{x}{\ln x - c}$
3. $(xe^{\frac{y}{x}} + y)dx - xdy = 0$ ans: $\ln|x| + e^{\frac{-y}{x}} = c$

3 - Linear

The equation of the form $\frac{dy}{dx}+p\cdot y=Q$ where P and Q are functions of only x or constant is called linear in y and $\frac{dy}{dx}$.

Let (I.F) $=e^{\int Pdx}$ be the integrating factor, then the general solution is

$y\cdot(I.f.)=\int(I.f.)\ Q\,.dx$

Ex.1: Solve $\frac{dy}{dx}-\frac{y}{x}=x\cdot e^{x}$

$P(x)=-\frac{1}{x},\quad Q(x)=x.e^{x}$

$(I.f.)=e^{\int\frac{-1}{x}dx}=e^{-\ln x}=\frac{1}{x}$

Solution is

$$y\cdot\frac{1}{x}=\int\frac{1}{x}\cdot xe^{x}\cdot dx$$

$$\frac{y}{x}=e^{x}+c$$

Ex.2:

Solve $\frac{dy}{dx}+x.y=x$

$P=x,\ Q=x$

$(I.f.)=e^{\int xdx}=e^{\frac{x^2}{2}}$

Solution is

$$y\cdot e^{\frac{x^2}{2}}=\int e^{\frac{x^2}{2}}\cdot x\cdot dx$$

$$y\cdot e^{\frac{x^2}{2}}=e^{\frac{x^2}{2}}+c\Rightarrow y=1+ce^{\frac{-x^2}{2}}\ \text{is the solution}$$

Exercise:

1. $\frac{dy}{dx}+2y=e^{-x}$ ans: $y=e^{-x}+ce^{-2x}$
2. $x\frac{dy}{dx}+3y=\frac{\sin x}{x^2}$ ans: $x^3y=c-\cos x$
3. xdy+ydx=ydy ans: $x=\frac{y}{2}+\frac{c}{y}$

Remark:

The equation $\frac{dx}{dy} + p \cdot x = Q$, where P and Q are functions of only y or constant, is said to be linear in x and $\frac{dx}{dy}$, and here (I.F.)= $e^{\int P dy}$ and the general solution is:

$$x \cdot (\text{I.f.}) = \int (\text{I.f.})\ Q \,.\, dy$$

Ex.3:Solve $e^{2y}dx+2(xe^{2y}-y)dy=0$

sol:

$$e^{2y}\frac{dx}{dy} + 2xe^{2y} - 2y = 0$$

$$\frac{dx}{dy} + 2x - 2ye^{-2y} = 0$$

$$\frac{dx}{dy} + 2x = 2ye^{-2y} \text{ is linear in x and } \frac{dx}{dy}$$

P=2, Q=$2ye^{-2y}$, I.F.= $e^{\int P(y)dy} = e^{2y}$

$$x \cdot (\text{I.f.}) = \int (\text{I.f.})\ Q \,.\, dy$$

$$e^{2y}x = \int e^{2y}\, 2ye^{-2y} dy$$

$$e^{2y}x = y^2 + c \Rightarrow e^{2y}x = y^2 + c \quad \text{is general solution}$$

4- Exact

The equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Ex.1:

Show that the following D.Eq. are exact D.Eq.

a) $(3x^2y + 2xy)dx + (x^3 + x^2 + 2y)dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 + 2x \quad , \quad \frac{\partial N}{\partial x} = 3x^2 + 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The D.Eq. is exact.

b) $[x\cos(x+y) + \sin(x+y)]dx + (x\cos(x+y)dy = 0$

$$\frac{\partial M}{\partial y} = -x\sin(x+y) + \cos(x+y)$$

$$\frac{\partial N}{\partial x} = -x\sin(x+y) + \cos(x+y)$$

$\therefore$ the D.Eq. is exact.

Ex.2: Is the D.Eq. $\frac{dy}{dx}=-\frac{(x^2+y^2)}{2xy}$ exact or not?

Sol.

$2xydy=-(x^2+y^2)dx$

$2xydy+(x^2+y^2)dx=0$

$\frac{\partial M}{\partial y}=2y \quad , \quad \frac{\partial N}{\partial x}=2y$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}, \quad \therefore$ the D.Eq. is exact

General Solution: the exact D.Eq., $M(x,y)dx+N(x,y)dy=0$, has a general solution f(x,y)=c where

$f(x,y)=\int Mdx+\int (terms\ \ in\ \ N\ \ \text{do not contains}\ \ x)dy$

Ex.3:

Solve the exact D.Eqs. $(3x^2y+2xy)dx+(x^3+x^2+2y)dy=0$

Sol.

$f(x,y)=\int (3x^2y+2xy)dx+\int 2y\text{dy}$

$=3y\cdot\frac{x^3}{3}+2y\cdot\frac{x^2}{2}+2\cdot\frac{y^2}{3}$

the solution is $x^3y+x^2y+y^2=c$

Ex.4:

Solve $(x+y)dx+(x+y^2)dy=0$

Sol.

$\frac{\partial M}{\partial y}=1 \quad , \quad \frac{\partial N}{\partial x}=1$

$\therefore$ the D.Eq. is exact

$f(x,y)=\int Mdx+\int (terms\ \ in\ \ N\ \ \text{do not contains}\ \ x)dy$

$=(x+y)dx+\int y^2\text{dy}$

$=\frac{x^2}{2}+xy+\frac{y^3}{3}$

the solution is $\frac{x^2}{2}+xy+\frac{y^3}{3}=c$

Exercise:

1. $(2+ye^{xy})dx+(xe^{xy}-2y)dy=0$ ans:$c=2x+e^{xy}-y^2$ ans: $2x+e^{xy}-y^2$
2. $(\tan x+\tan y)dy+(y\sec^2x+\sec x\tan x)dx=0$ ans: $c=y\tan x-\ln\cos y+\sec x$
3. $(2xy+y^2)dx+(x^2+2xy-y)dy=0$ ans: $x^2y+y^2x-y^2/2=c$

Problems:

Solve the following differential equations:

1- $y\ln y dx + (1 + x^2)dy = 0$

2- $e^{x+2y}dy - e^{y-2x}dx = 0$

3- $(2x + y)dx + (x - 2y)dy = 0$

4- $x\, dy = (y + x\cos^2(\frac{y}{x}))dx = 0$

5- $x(\ln y - \ln x)dy = y(1 + \ln y - \ln x)dx$

6- $x\, dy + (2y - x^2 - 1)dx = 0$

7- $\cos y\, dx + (x\,\text{siny - cos}^2 y)dy = 0$

8- $(1 + y^2)dx + (2xy + y^2 + 1)dy = 0$

9- $(e^x + \ln y)dx + (\frac{x + y}{y})dy = 0$

10- $x(1 + e^y)dx + \frac{1}{2}(x^2 + y^2)e^y dy = 0$

References:

1- Calculus & Analytic Geometry (Thomas).
2- Calculus (Haward Anton).
3- Advanced Mathematics for Engineering Studies **(أ. رياض احمد عزت)**
4- Modern Introduction Differential Equations, Schaum's Outline Series.